NECESSARY AND SUFFICIENT CONDITIONS FOR THE INTERCHANGE OF LIMIT AND SUMMATION IN THE CASE OF SEQUENCES OF INFINITE SERIES OF A CERTAIN TYPE.

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Double sequences and series have been discussed by Pringsheim,* London,† and others. They have treated incidentally the question of interchange of iterated limits. An interchange of limit and infinite summation, although to some extent a question of interchange of limits, is a distinct problem in that summation and limit are different operations. The theorems of this paper derive necessary and sufficient conditions for the interchange of limit in the case of a special type of series, practically that of series of positive terms.

THEOREM I. *Suppose a double sequence of numbers* x_{np} $(n = 1, 2, \cdots; p = 1, 2, \cdots)$ *such that* $\Sigma_p|x_{np}|$ ‡ *is convergent for every* n. *Suppose also* $L_n x_{np} = x_p$ *for every* p. *Then a necessary and sufficient condition that* $\Sigma_p|x_p|$ *converge and* $L_n\Sigma_p|x_{np}| = \Sigma_p|x_p|$ *is that the series* $\Sigma_p|x_{np}|$ *be uniformly convergent.*

That the condition is *sufficient* even in case the absolute value signs be dropped throughout is well known, a consequence of the theorem on the interchange of double limits.§

On the other hand the condition is *necessary*. Since $\Sigma_p|x_p|$ converges we have for every $e > 0$ a p_e such that if $P \geqq p_e$ then:

$$\sum_{p=P}^{\infty} |x_p| \leqq e/2.$$

Take a particular value of P, say $p_1 = p_e$. Then from

$$\underset{n}{L} \sum_{p} |x_{np}| = \sum_{p} |x_p| \qquad \text{follows} \qquad \underset{n}{L} \sum_{p=p_1}^{\infty} |x_{np}| = \sum_{p=p_1}^{\infty} |x_p|,$$

i. e., for every e there exists an n_e such that if $n \geqq n_e$ we have:

$$\left| \sum_{p=p_1}^{\infty} |x_{np}| - \sum_{p=p_1}^{\infty} |x_p| \right| \leqq e/2.$$

* Pringsheim, Muench. Ber. (1897), pp. 101–152; Math. Ann., 53 (1900), pp. 289–321.

† London, Math. Ann., 53 (1900), pp. 322-370.

‡ Throughout this paper we shall designate $\sum_{p=1}^{\infty}$ by $\sum_{p}$ and $\underset{n=\infty}{L}$ by $\underset{n}{L}$.

§ Cf., for instance, Hobson, Theory of Functions, p. 466.

Then if $P \gtreqless p_1 = p_\epsilon$ and $n \gtreqless n_e$, we have:

$$\sum_{p=P}^{\infty} |x_{np}| \leqq \sum_{p=p_1}^{\infty} |x_{np}| \lesseqgtr \sum_{p=p_1}^{\infty} |x_p| + e/2 \lesseqgtr e/2 + e/2 = e.$$

Since there will be a finite number of values of n less than n_e we have: for every $e > 0$ there exists a p_e' such that for $P \gtreqless p_e'$ we have:

$$\sum_{p=P}^{\infty} |x_{np}| \lesseqgtr e, \qquad n < n_e.$$

Hence if p_e'' is the greater of p_e and p_e' we have for every n and for every $e > 0$ there exists a p_e'' such that for $P \gtreqless p_e''$ we have:

$$\sum_{p=P}^{\infty} |x_{np}| \lesseqgtr e;$$

which is the uniformity of convergence desired.

COROLLARY. *If $m > 0$, $\Sigma_p|x_{np}|^m$ is convergent for every n and $L_n x_{np} = x_p$ for every p, then a necessary and sufficient condition that $\Sigma_p|x_p|^m$ be convergent and $L_n\Sigma_p|x_{np}|^m = \Sigma_p|x_p|^m$ is that $\Sigma_p|x_{np}|^m$ be uniformly convergent.*

THEOREM II. *Suppose $m > 0$, $\Sigma_p|x_{np}|^m$ convergent for every n, and $L_n x_{np} = x_p$ for every p. Then a necessary and sufficient condition that $\Sigma_p|x_p|^m$ be convergent and $L_n\Sigma_p|x_{np}|^m = \Sigma_p|x_p|^m$, is that $L_n\Sigma_p|x_{np} - x_p|^m = 0$.*

To prove this theorem we make use of the following inequality *

$$\left[\sum_{p=1}^{n} |a_p + b_p|^m\right]^k \lesseqgtr \left[\sum_{p=1}^{n} |a_p|^m\right]^k + \left[\sum_{p=1}^{n} |b_p|^m\right]^k, \tag{1}$$

where $m > 0$, and $k = 1/m$ if $m > 1$ and $k = 1$ if $m < 1$. This inequality may be extended to n infinite if the series on the right hand side are convergent. We shall refer to this extension as inequality (1′).

The condition of the theorem is *necessary*. Since $\Sigma_p|x_p|^m$ is convergent and $L_n\Sigma_p|x_{np}|^m = \Sigma_p|x_p|^m$, we have by the corollary above that $\Sigma_p|x_{np}|^m$ are uniformly convergent, i. e., for every $e > 0$ there exists a p_e such that if $P \gtreqless p_\epsilon$, we have:

$$\sum_{p=P}^{\infty} |x_{np}|^m \lesseqgtr e^{1/k} \qquad \text{and} \qquad \sum_{p=P}^{\infty} |x_p|^m \lesseqgtr e^{1/k}.$$

Take $P = p_1$ fixed. For $p = 1, 2, \cdots, p_1$, we have, since $L_n x_{np} = x_p$ for every p: for every $e > 0$ there exists an n_ϵ such that if $n \gtreqless n_\epsilon$ we have:

$$|x_{np} - x_p|^m \lesseqgtr (e/p_1)^{1/k}. \tag{2}$$

* Cf. Riess, Math. Ann., vol. 69 (1910), p. 455.

Using the inequality (1′) and the fact that $k \leqq 1$ for every m, we have:

$$\left[\sum_p |x_{np} - x_p|^m\right]^k \leqq \sum_{p=1}^{p_1}\left[|x_{np} - x_p|^m\right]^k + \left[\sum_{p=p_1}^{\infty}|x_{np} - x_p|^m\right]^k$$

$$\leqq e + \left[\sum_{p=p_1}^{\infty}{}' |x_{np}|^m\right]^k + \left[\sum_{p=p_1}^{\infty} |x|^m\right]^k \leqq 3e,$$

i. e., for every $e > 0$ there exists an n_e, viz., the n_e needed for inequality (2), such that if $n \geqq n_e$, we have:

$$\left[\sum_{p=1}^{\infty} |x_{np} - x_p|^m\right]^k \leqq e,$$

in other words:

$$\underset{n}{L}\sum_p |x_{np} - x_p|^m = 0.$$

The condition is *sufficient.** Since $L_n\Sigma_p|x_{np} - x_p|^m = 0$, for every $e > 0$ there will exist an n_e such that if $n \geqq n_e$ we have:

(3) $$\sum_p |x_{np} - x_p|^m \leqq e.$$

Let $x_{np} - x_p = e_{np}$. Then $x_p = x_{np} + e_{np}$ and $x_{np} = x_p - e_{np}$. Applying inequality (1′), we obtain:

$$[\sum_p |x_p|^m]^k = [\sum_p |x_{np} + e_{np}|^m]^k \leqq [\sum_p |x_{np}|^m]^k + [\sum_p |e_{np}|^m]^k$$

and

$$[\sum_p |x_{np}|^m]^k = [\sum_p |x_p - e_{np}|^m]^k \leqq [\sum_p |x_p|^m]^k + [\sum_p |e_{np}|^m]^k.$$

From the first of these inequalities we conclude that $\sum_p |x_p|^m$ is convergent.† From the two inequalities taken together and the condition $n \geqq n_e$, which gives us inequality (3), we have:

$$[\sum_p |x_{np}|^m]^k - e^k \leqq [\sum_p |x_p|^m]^k \leqq [\sum_p |x_{np}|^m]^k + e^k$$

and so

$$\underset{n}{L}\sum_p |x_{np}|^m = \sum_p |x_p|^m.$$

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* For $m = 2$ this is a consequence of a theorem due to Hilbert, Goett. Nach. Math. Phys. Klasse (1906), p. 177.

† Also a consequence of Moore, General Analysis (Yale Coll. Lect.) §16, p. 38.